

# BFFT quantization with nonlinear constraints

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## Abstract

We consider the method due to Batalin, Fradkin, Fradkina, and Tyutin (BFFT) that makes the conversion of second-class constraints into first-class ones for the case of nonlinear theories. We first present a general analysis of an attempt to simplify the method, showing the conditions that must be fulfilled in order to have first-class constraints for nonlinear theories but that are linear in the auxiliary variables. There are cases where this simplification cannot be done and the full BFFT method has to be used. However, in the way the method is formulated, we show with details that it is not practicable to be done. Finally, we speculate on a solution for these problems.

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# 1 Introduction

The quantization method due to Batalin, Fradkin, Fradkina, and Tyutin [1, 2] has as main purpose the conversion of second-class constraints into first-class ones. This is achieved with the aid of auxiliary variables that extend the phase space in a convenient way. After that, we have a gauge invariant system which matches the original theory when the so-called unitary gauge is chosen.

The BFFT method is quite elegant and operates systematically. The obtainment of first-class constraints is done in an iterative process. The first correction of the constraints is linear in the auxiliary variables, the second one is quadratic, and so on. In the case of systems with just linear constraints, we obtain that just linear corrections are enough to make them first-class. Here, we mention that the method is equivalent to express the dynamic quantities by means of shifted coordinates [3].

However, for systems with nonlinear constraints, the iterative process may be go beyond the first correction. At this point resides the problem we would like to address. The question is that the first step of the process does not precisely fix the solution that shall be used in the next steps. In recent papers, Banerjee et al. [4] have shown that an intelligent choice among the solutions displayed at the first stage for the  $O(N)$  nonlinear sigma-model and the  $CP^{N-1}$  can stop the iterative process there, avoiding the possible inconveniences that could be found into the next steps. This is a kind of simplification for the use of the method, but unfortunately it cannot be applied for all cases.

In the first part of this paper, we do a general analysis of the conditions that makes possible the obtainment of linear first-class constraints (in the auxiliary variables) for nonlinear theories. We then show why the choice made by Banerjee et al. has worked on. We concentrate on the  $O(N)$  nonlinear sigma model (the  $CP^{N-1}$  can be verified in a similar way). After that, we deal with the supersymmetric version of the  $O(N)$  nonlinear sigma model and show that it is also possible to have a linear solution in the auxiliary variables for this case too, but being nonlocal. We deserve Sec. 3 for these developments and use Sec. 2 for a brief review of the BFFT formalism in order to become clear the problem we would like to address our attention. In Sec. 4, we consider some cases where these choices are not or cannot be done and it is necessary to go to the next order of the iterative process. However, in a example we shall discuss, there occurs an additional problem, besides the one related to the choice of the solution that shall be used in the next steps: we shall see that nor all solutions of the first process can lead to a solution in the second one. If fact, we were just able to obtain a very particular solution of the first step that could lead to a solution in the second step. This fact show that the method is not feasible to be used in the general case with nonlinear constraints. We left an Appendix to contain some details of calculation and Sec. 5 for some concluding remarks.

## 2 Brief review of the BFFT formalism

Let us consider a system described by a Hamiltonian  $H_0$  in a phase-space  $(q^i, p_i)$  with  $i = 1, \dots, N$ . Here we suppose that the coordinates are bosonic (extension to include fermionic degrees of freedom and to the continuous case can be done in a straightforward way). Let us also suppose that there just exist second-class constraints. Denoting them by  $T_a$ , with  $a = 1, \dots, M < 2N$ , we have

$$\{T_a, T_b\} = \Delta_{ab}, \quad (2.1)$$

where  $\det(\Delta_{ab}) \neq 0$ .

As it was said, the general purpose of the BFFT formalism is to convert second into first-class constraints. This is achieved by introducing auxiliary canonical variables, one for each second-class constraint (the connection between the number of second-class constraints and the new variables in one-to-one is to keep the same number of the physical degrees of freedom in the resulting extended theory). We denote these auxiliary variables by  $\eta^a$  and assume that they have the following general structure

$$\{\eta^a, \eta^b\} = \omega^{ab}, \quad (2.2)$$

where  $\omega^{ab}$  is a constant quantity with  $\det(\omega^{ab}) \neq 0$ . The obtainment of  $\omega^{ab}$  is discussed in what follows. It is embodied in the calculation of the resulting first-class constraints that we denote by  $\tilde{T}_a$ . Of course, these depend on the new variables  $\eta^a$ , namely

$$\tilde{T}_a = \tilde{T}_a(q, p; \eta), \quad (2.3)$$

and satisfy the boundary condition

$$\tilde{T}_a(q, p; 0) = T_a(q, p). \quad (2.4)$$

Another characteristic of these new constraints is that they are assumed to be strongly involutive, i.e.

$$\{\tilde{T}_a, \tilde{T}_b\} = 0. \quad (2.5)$$

We emphasize that this is a characteristic of the BFFT formalism, i.e., the obtained first-class constraints are always supposed to satisfy an Abelian algebra.

The solution of (2.5) can be achieved by considering  $\tilde{T}_a$  expanded as

$$\tilde{T}_a = \sum_{n=0}^{\infty} T_a^{(n)}, \quad (2.6)$$

where  $T_a^{(n)}$  is a term of order  $n$  in  $\eta$ . Compatibility with the boundary condition (2.4) requires

$$T_a^{(0)} = T_a. \quad (2.7)$$

The replacement of (2.6) into (2.5) leads to a set of equations, one for each coefficient of powers of  $\eta$ . We list some of them below

$$\{T_a^{(0)}, T_b^{(0)}\}_{(q,p)} + \{T_a^{(1)}, T_b^{(1)}\}_{(\eta)} = 0, \quad (2.8)$$

$$\{T_a^{(0)}, T_b^{(1)}\}_{(q,p)} + \{T_a^{(1)}, T_b^{(0)}\}_{(q,p)} + \{T_a^{(1)}, T_b^{(2)}\}_{(\eta)} + \{T_a^{(2)}, T_b^{(1)}\}_{(\eta)} = 0, \quad (2.9)$$

$$\begin{aligned} \{T_a^{(0)}, T_b^{(2)}\}_{(q,p)} + \{T_a^{(1)}, T_b^{(1)}\}_{(q,p)} + \{T_a^{(2)}, T_b^{(0)}\}_{(q,p)} + \{T_a^{(1)}, T_b^{(3)}\}_{(\eta)} \\ + \{T_a^{(2)}, T_b^{(2)}\}_{(\eta)} + \{T_a^{(3)}, T_b^{(1)}\}_{(\eta)} = 0, \end{aligned} \quad (2.10)$$

$\vdots$

The notation  $\{, \}_{(q,p)}$  and  $\{, \}_{(\eta)}$  represent the parts of the Poisson bracket relative to the variables  $(q, p)$  and  $(\eta)$ , respectively.

The BFFT method establishes that equations above are used iteratively in the obtainment of the corrections  $T^{(n)}$  ( $n \geq 1$ ). Equation (2.8) shall give us  $T^{(1)}$ . With this result and (2.9), one calculates  $T^{(2)}$ , and so on. To calculate  $T^{(1)}$  (that is linear in  $\eta$ ), we may write

$$T_a^{(1)} = X_{ab}(q, p) \eta^b. \quad (2.11)$$

Introducing this expression into (2.8) and using the boundary condition (2.4), as well as (2.1) and (2.2), we get

$$\Delta_{ab} + X_{ac} \omega^{cd} X_{bd} = 0. \quad (2.12)$$

We notice that this equation does not give  $X_{ab}$  univocally, because it also contains the still unknown  $\omega^{ab}$ . What we usually do is to choose  $\omega^{ab}$  in such a way that the new variables are unconstrained. It might be opportune to mention that sometimes it is not possible to make a choice like that [5]. In this case, the new variables are constrained. In consequence, the consistency of the method requires an introduction of other new variables in order to transform these constraints also into first-class. This may lead to an endless process. But it is important to emphasize that  $\omega^{ab}$  can be fixed anyway.

However, even when one fixes  $\omega^{ab}$  it is still not possible to obtain a univocal solution for  $X_{ab}$ . Let us check this point. Since we are only considering bosonic coordinates<sup>1</sup>,  $\Delta_{ab}$  and  $\omega^{ab}$  are antisymmetric quantities. So, expression (2.12) compactly represents  $M(M-1)/2$  independent equations. On the other hand, there

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<sup>1</sup>This problem also exists in the fermionic sector.

is no prior symmetry involving  $X_{ab}$  and they consequently represent a set of  $M^2$  independent quantities.

In the case where  $X_{ab}$  do not depend on  $(q, p)$ , it is easily seen that  $T_a + T_a^{(1)}$  is already strongly involutive for any choice we make and we are succeed in obtaining  $\tilde{T}_a$ . If this is not so, the usual procedure is to introduce  $T_a^{(1)}$  into equation (2.9) to calculate  $T_a^{(2)}$ , and so on. At this point resides the problem we had mentioned. We do not know a priori what is the best choice we can make to go from one step to another. More than that, we are going to show an example where solutions of the first step do not necessary lead to a solution in the second one.

What Banerjee et al. have done is a kind of simplification of the BFFT method that consists to look for a convenient set of coefficients, among the options contained in eq. (2.12), in such a way that the first-class constraints algebra could be obtained in the first stage of the process. They were succeeded in the particular case of the  $O(N)$  nonlinear sigma model and  $CP^{N-1}$  without Lagrange multiplier fields. In order to see in a general way for what conditions this choice can be done, we the impose that the linear constraints (in the auxiliary variables)

$$\tilde{T}_a = T_a + X_{ab} \eta^b \quad (2.13)$$

satisfy the strong (Abelian) algebra (2.5). Besides eq. (2.12), there remains the following equations to be satisfied

$$\{T_a, X_{bc}\} + \{X_{ac}, T_b\} = 0, \quad (2.14)$$

$$\{X_{ac}, X_{bd}\} + \{X_{ad}, X_{bc}\} = 0. \quad (2.15)$$

We notice that when  $X_{ab}$  do not depend on the phase space coordinates, the equations above are satisfied trivially. However, when this is not so, we have more equations than variables and each case has to be verified in a separate way. We are going to discuss in the next section why the  $O(N)$  nonlinear sigma model can be considered as a particular example where eqs. (2.14) and (2.15) are verified.

### 3 Simplified use of the BFFT formalism

Let us concentrate here in the  $O(N)$  nonlinear sigma-model. It is described by the Lagrangian

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi^A \partial^\mu \phi^A + \frac{1}{2} \lambda (\phi^A \phi^A - 1), \quad (3.1)$$

where the index  $A$  is related to the  $O(N)$  symmetry group. Let us obtain the constraints [6]. The canonical momenta are

$$\pi^A = \frac{\partial \mathcal{L}}{\partial \dot{\phi}^A} = \dot{\phi}^A, \quad (3.2)$$

$$p = \frac{\partial \mathcal{L}}{\partial \dot{\lambda}} = 0. \quad (3.3)$$

We notice that (3.3) is a primary constraint. In order to look for secondary constraints we construct the Hamiltonian

$$\begin{aligned} \mathcal{H} &= \pi^A \dot{\phi}^A + p \dot{\lambda} - \mathcal{L} + \xi p, \\ &= \frac{1}{2} \pi^A \pi^A + \frac{1}{2} \phi^{A'} \phi^{A'} - \frac{1}{2} \lambda (\phi^A \phi^A - 1) + \tilde{\xi} p, \end{aligned} \quad (3.4)$$

where the velocity  $\dot{\lambda}$  was absorbed in the Lagrange multiplier  $\tilde{\xi}$  by the redefinition  $\tilde{\xi} = \xi + \dot{\lambda}$ . The consistency condition for the constraint  $p = 0$  leads to another constraint

$$\phi^A \phi^A - 1 = 0. \quad (3.5)$$

At this stage, we have two options. The first one is to introduce the constraint above into the Hamiltonian by means of another Lagrange multiplier. The result is

$$\begin{aligned} \mathcal{H} &= \frac{1}{2} \pi^A \pi^A + \frac{1}{2} \phi^{A'} \phi^{A'} - \frac{1}{2} \lambda (\phi^A \phi^A - 1) + \tilde{\xi} p + \zeta (\phi^A \phi^A - 1), \\ &= \frac{1}{2} \pi^A \pi^A + \frac{1}{2} \phi^{A'} \phi^{A'} + \tilde{\xi} p + \tilde{\zeta} (\phi^A \phi^A - 1). \end{aligned} \quad (3.6)$$

The field  $\lambda$  was also absorbed by the Lagrange multiplier  $\tilde{\zeta}$  ( $\tilde{\zeta} = \zeta - \frac{1}{2} \lambda$ ). The consistency condition for the constraint (3.5) leads to another more constraint

$$\phi^A \pi^A = 0. \quad (3.7)$$

We mention that the consistency condition for this constraint will give us the Lagrange multiplier  $\tilde{\zeta}$  and no more constraints. Since we have absorbed the field  $\lambda$ , its momentum  $p$  does not play any role in the theory and we can disregard it by using the constraint relation (3.3) in a strong way. So the constraints of the theory in this case are

$$\begin{aligned} T_1 &= \phi^A \phi^A - 1, \\ T_2 &= \pi^A \phi^A. \end{aligned} \quad (3.8)$$

The other option we could have followed was to keep the Lagrange multiplier  $\lambda$  in the theory. To do this, we consider that the constraint (3.5) is already in the Hamiltonian due to the presence of the term  $-\frac{\lambda}{2}(\phi^A\phi^A - 1)$ . So, instead of the Hamiltonian (3.6) we use the previous one given by (3.4) in order to verify the consistency condition of the constraint (3.5). It is easily seen that the constraint (3.7) is obtained again, and the consistency condition for it leads to a new constraint

$$\pi^A\pi^A + \phi^A\phi^{A''} + \lambda\phi^A\phi^A = 0. \quad (3.9)$$

No more constraints are obtained. So, according to the second option, the constraints of the theory are

$$\begin{aligned} T_1 &= \phi^A\phi^A - 1, \\ T_2 &= \pi^A\phi^A, \\ T_3 &= p, \\ T_4 &= \pi^A\pi^A + \lambda\phi^A\phi^A + \phi^A\phi^{A''}. \end{aligned} \quad (3.10)$$

We mention that the set given by (3.10) is an example where it is necessary to go to higher order of the iterative process of the BFFT method. We shall discuss this example in the next section. Let us concentrate here on the first set of constraints (3.8). The quantity  $\Delta_{ab}$ , defined by eq. (2.1), can be directly calculated. The result is

$$\Delta_{ab} = 2 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \phi^A\phi^A \delta(x-y). \quad (3.11)$$

At this point we could be tempted to use back the constraint  $T_1$  in order to simplify the expression above. This cannot be done because expressions (2.8)-(2.10) have to be verified in a strong way.

Let us extend the phase space by introducing two new variables  $(\eta^1, \eta^2)$  and consider that  $\eta^2$  is the canonical momentum conjugate to  $\eta^1$ . Consequently, expression (2.12) gives us just one equation

$$X_{12}X_{21} - X_{11}X_{22} = 2\phi^A\phi^A \delta(x-y), \quad (3.12)$$

where there are four quantities to be fixed. The simplified use of the BFFT formalism requires that these quantities also satisfy expressions (2.14) and (2.15), what correspond to five more equations.

Eventhough there are more equations (six) than variables (four), the system presents solution, as has been pointed out by Banerjee et al. but considering a different analysis. The arguments we use are the following. Looking at eq. (3.12), we notice that at least one of the variables must contain the field  $\phi^A$ . This is not

a problem to satisfy expressions (2.15). However, concerning the ones given (2.14) we need some care because the constraint  $T_2$  contains the momentum  $\pi^A$ . The way we have to satisfy expression (2.14) for  $a = 2$  (the index where  $T_2$  will appear) is to use the fact that (2.14) is antisymmetric in the indices  $a$  and  $b$ . So, if we choose the coefficients  $X_{2c}$  to contain the fields  $\phi^A$ , equations (2.14) will be automatically verified. The Banerjee et al. choice was (up to some multiplicative factors)

$$\begin{aligned} X_{11} &= 2\delta(x-y), \\ X_{22} &= -\phi^A\phi^A\delta(x-y), \\ X_{12} &= X_{21} = 0, \end{aligned} \tag{3.13}$$

that leads to the following set of first-class constraints

$$\begin{aligned} \tilde{T}_1 &= \phi^A\phi^A - 1 + 2\eta^1, \\ \tilde{T}_2 &= \phi^A\pi^A - \phi^A\phi^A\eta^2. \end{aligned} \tag{3.14}$$

Another general choice could be to use the coefficients  $X_{21}$  to contain  $\phi^A$ , that is

$$\begin{aligned} X_{12} &= 2\delta(x-y), \\ X_{21} &= \phi^A\phi^A\delta(x-y), \\ X_{11} &= X_{22} = 0. \end{aligned} \tag{3.15}$$

The second choice is nothing other than the interchange of  $\eta^1$  and  $\eta^2$  in equations (3.14), accompanied by a convenient change of signs. In any case, both solutions close in a Abelian algebra like (2.5).

An interesting point is that the simplified use of the BFFT formalism can also be applied to the supersymmetric case. The constraints in this case are (still without Lagrange multipliers)

$$\begin{aligned} T_1 &= \phi^A\phi^A - 1, \\ T_2 &= \phi^A\pi^A, \\ T_{3\alpha} &= \phi^A\psi_\alpha^A, \end{aligned} \tag{3.16}$$

where the canonical momentum conjugate to  $\psi_\alpha^A$  is a constraint relation that can be eliminated by using the Dirac bracket <sup>2</sup> [6]

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<sup>2</sup>We could have used auxiliary variables to convert these constraints into first-class too. But this would be a trivial operation that would not lead to any new significant result.



$$\{\psi_\alpha^A(x), \psi_\beta^B(y)\} = -i \delta_{\alpha\beta} \delta^{AB} \delta(x-y). \quad (3.17)$$

We thus have

$$\begin{aligned} \Delta_{12} &= 2\phi^A \phi^A \delta(x-y), \\ \Delta_{23} &= -\phi^A \psi_\alpha^A \delta(x-y), \\ \Delta_{3\alpha 3\beta} &= -i \delta_{\alpha\beta} \phi^A \phi^A \delta(x-y). \end{aligned} \quad (3.18)$$

The remaining quantities vanish. We extend the phase-space by introducing the coordinates:  $\eta_1$ ,  $\eta_2$  and  $\chi_\alpha$  and consider that they satisfy the fundamental relations

$$\begin{aligned} \{\eta_1, \eta_2\} &= \delta(x-y), \\ \{\chi_\alpha, \chi_\beta\} &= -i \delta_{\alpha\beta} \delta(x-y). \end{aligned} \quad (3.19)$$

Let us verify if it is also possible to have a solution for equations (2.12), (2.14) and (2.15). These have to be slightly modified by virtue of the presence of fermionic constraints, namely

$$\Delta_{ab} + (-1)^{(\epsilon_b+1)\epsilon_a} X_{ac} \omega^{cd} X_{bd} = 0, \quad (3.20)$$

$$\{T_a, X_{bc}\} + (-1)^{\epsilon_b \epsilon_c} \{X_{ac}, T_b\} = 0, \quad (3.21)$$

$$\{X_{ac}, X_{bd}\} + \{X_{ad}, X_{bc}\} = 0, \quad (3.22)$$

where  $\epsilon_a = 0$  for  $a = 1, 2$  (bosonic constraints) and  $\epsilon_a = 1$  otherwise (fermionic constraints). The expression (3.20) yields the following set of equations

$$X_{12}X_{21} - X_{11}X_{22} - iX_{13\alpha}X_{23\alpha} = 2\phi^A \phi^A \delta(x-y), \quad (3.23)$$

$$X_{21}X_{3\alpha 2} - X_{22}X_{3\alpha 1} - iX_{23\beta}X_{3\alpha 3\beta} = \phi^A \psi_\alpha^A \delta(x-y), \quad (3.24)$$

$$X_{3\alpha 1}X_{3\beta 2} - X_{3\alpha 2}X_{3\beta 1} - iX_{3\alpha 3\gamma}X_{3\beta 3\gamma} = i\delta_{\alpha\beta} \phi^A \phi^A \delta(x-y), \quad (3.25)$$

$$X_{11}X_{3\alpha 2} - X_{12}X_{3\alpha 1} - iX_{13\beta}X_{3\alpha 3\beta} = 0. \quad (3.26)$$

From equation (3.25) we are forced to conclude that

$$X_{3\alpha 3\beta}^A = i\delta_{\alpha\beta} \sqrt{\phi^A \phi^A} \delta(x-y), \quad (3.27)$$

and a careful analysis of the remaining equations permit us to infer that a solution that is also compatible with (3.21) and (3.22) is given by

$$X_{11} = 2\delta(x-y), \quad (3.28)$$

$$X_{22} = -\phi^A\phi^A\delta(x-y), \quad (3.29)$$

$$X_{23\alpha} = \frac{\phi^A\psi_\alpha^A}{\sqrt{\phi^B\phi^B}}\delta(x-y). \quad (3.30)$$

The remaining coefficients are zero. The choice given by equations (3.27)-(3.30) leads to the following set of first-class constraints

$$\begin{aligned} \tilde{T}_1 &= \phi^A\phi^A - 1 + 2\eta^1, \\ \tilde{T}_2 &= \phi^A\pi^A - \phi^A\phi^A\eta^2 + \frac{\phi^A\psi_\alpha^A\chi_\alpha}{\sqrt{\phi^B\phi^B}}, \\ \tilde{T}_{3\alpha} &= \phi^A\psi_\alpha^A + i\sqrt{\phi^A\phi^A}\chi_\alpha. \end{aligned} \quad (3.31)$$

The above result shows us that it is also possible to obtain a solution, eventhough nonlocal, for the simplified version of the BFFT method when applied to the supersymmetric nonlinear sigma model.

## 4 Beyond the first interative step

In order to have a better understanding of the problem we are going to deal with, let us first consider the case without Lagrange multipliers, and take a different choice of the one given by Banerjee et al. For example, let us suppose that instead of solutions (3.13) or (3.15), we take

$$\begin{aligned} X_{11} &= \phi^A\phi^A\delta(x-y), \\ X_{22} &= -2\delta(x-y), \\ X_{12} &= X_{21} = 0. \end{aligned} \quad (4.1)$$

For this choice, we have

$$\begin{aligned} T_1^{(1)} &= \phi^A\phi^A\eta^1, \\ T_2^{(1)} &= -2\eta^2. \end{aligned} \quad (4.2)$$

We easily see that eq. (2.14) is not more verified and consequently the quantities  $T_1 + T_1^{(1)}$  and  $T_2 + T_2^{(1)}$  do not form first-class constraints anymore. It is then

necessary to go to the second step of the BFFT method, what corresponds to use the equation (2.9), namely

$$\{T_a, T_b^{(1)}\}_{(\phi, \pi)} + \{T_a^{(1)}, T_b\}_{(\phi, \pi)} + \{T_a^{(1)}, T_b^{(2)}\}_{(\eta)} + \{T_a^{(2)}, T_b^{(1)}\}_{(\eta)} = 0. \quad (4.3)$$

The combination of expressions (3.8), (4.2), and (4.3) permit us to infer the following solution for  $T_a^{(2)}$

$$\begin{aligned} T_1^{(2)} &= \eta^2 \eta^2, \\ T_2^{(2)} &= -2\eta^1 \eta^2. \end{aligned} \quad (4.4)$$

We then notice that the quantities  $\tilde{T}_a = T_a + T_a^{(1)} + T_a^{(2)}$ , namely

$$\begin{aligned} \tilde{T}_1 &= \phi^A \phi^A - 1 + \phi^A \phi^A \eta^1 + \eta^2 \eta^2, \\ \tilde{T}_2 &= \phi^A \pi^A - 2\eta^2 - 2\eta^1 \eta^2 \end{aligned} \quad (4.5)$$

are first-class constraints and close in an Abelian algebra. Thus, without the choice made by Banerjee et al., it is necessary to go to the second order of the iterative process. The obtained first-class constraints, for this particular version of the  $O(N)$  nonlinear sigma-model, are quadratic in the auxiliary variables.

The same analysis could also have been done for the supersymmetric case (without Lagrange multipliers). We just mention that the set of first-class constraints is given by

$$\begin{aligned} \tilde{T}_1 &= \phi^A \phi^A (1 + \eta^1) - 1, \\ \tilde{T}_2 &= \phi^A \pi^A - 2\eta^2 - 2\eta^1 \eta^2 + \frac{\phi^A \psi_\alpha^A \chi_\alpha}{\sqrt{\phi^B \phi^B}}, \\ \tilde{T}_{3\alpha} &= \phi^A \psi_\alpha^A + i \sqrt{\phi^A \phi^A} \chi_\alpha. \end{aligned} \quad (4.6)$$

Let us now consider the more general version of the  $O(N)$  nonlinear sigma model, which corresponds to the set of constraints given by expressions (3.10). First we write down the quantities  $\Delta_{ab}$ . The nonvanishing ones are

$$\begin{aligned} \Delta_{12}(x, y) &= 2\phi^A \phi^A \delta(x - y), \\ \Delta_{14}(x, y) &= 4\phi^A \pi^A \delta(x - y), \\ \Delta_{24}(x, y) &= 2(\pi^A \pi^A - \lambda \phi^A \phi^A - \phi^A \phi^{A''}) \delta(x - y) \\ &\quad - 2\phi^A \phi^{A'} \delta'(x - y) - \phi^A \phi^A \delta''(x - y), \\ \Delta_{34}(x, y) &= -\phi^A \phi^A \delta(x - y), \end{aligned} \quad (4.7)$$

where prime always means derivative with respect to  $x$ . Here, we extend the phase-space by introducing four coordinates  $\eta^a$  and consider that  $\eta^3$  and  $\eta^4$  are the canonical momenta conjugated to  $\eta^1$  and  $\eta^2$ , respectively. Thus, the matrix  $(\omega^{ab})$  reads

$$(\omega^{ab}) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \delta(x-y). \quad (4.8)$$

The combination of (2.12), (4.7) and (4.8) gives us the set of equations

$$\begin{aligned} X_{11}X_{23} + X_{12}X_{24} - X_{13}X_{21} - X_{14}X_{22} &= -2\phi^A\phi^A\delta, \\ X_{11}X_{33} + X_{12}X_{34} - X_{13}X_{31} - X_{14}X_{32} &= 0, \\ X_{11}X_{43} + X_{12}X_{44} - X_{13}X_{41} - X_{14}X_{42} &= -4\phi^A\pi^A\delta, \\ X_{21}X_{33} + X_{22}X_{34} - X_{23}X_{31} - X_{24}X_{32} &= 0, \\ X_{21}X_{43} + X_{22}X_{44} - X_{23}X_{41} - X_{24}X_{42} &= -2(\pi^A\pi^A - \lambda\phi^A\phi^A - \phi^A\phi^{A''})\delta \\ &\quad + 2\phi^A\phi^{A'}\delta' + \phi^A\phi^A\delta'', \\ X_{31}X_{43} + X_{32}X_{44} - X_{33}X_{41} - X_{34}X_{42} &= \phi^A\phi^A\delta. \end{aligned} \quad (4.9)$$

where  $\delta$ ,  $\delta'$ ,  $\delta''$  stand for simplified notations of the delta function and its derivatives. We easily observe that the arguments used in the case without the Lagrange multiplier fields cannot be applied here. Now, the coefficients  $X_{ab}$  contain momenta and it is not trivial that eq. (2.15) can be verified. Further, we also have momenta in more than one constraint and this consequently means that we cannot fix one of eqs. (2.14) without spoiling the others. This is an example where it actually necessary to go to higher order of the iterative process of the BFFT method.

However, an interesting and, in some sense, unexpected fact occurs: nor all solution obtained in the first stage leads to a solution in the second one. We shall discuss this point with details in the Appendix. Let us just write down a convenient solution of the first step of the method (that leads to a solution into the second step)

$$\begin{aligned} X_{11} &= -2\phi^A\phi^A\delta, & X_{12} &= 0, & X_{13} &= 0, & X_{14} &= 4\phi^A\pi^A\delta, \\ X_{21} &= 0, & X_{22} &= 0, & X_{23} &= \delta, & X_{24} &= 2\pi^A\pi^A\delta, \\ X_{31} &= 0, & X_{32} &= 0, & X_{33} &= 0, & X_{34} &= -\phi^A\phi^A\delta \\ X_{41} &= -2\lambda\phi^A\phi^A\delta - 2\phi^A\phi^{A''}\delta - 2\phi^A\phi^{A'}\delta' - \phi^A\phi^A\delta'', \\ X_{42} &= \delta, & X_{43} &= 0, & X_{44} &= -4\phi^{A'}\pi^A\delta', \end{aligned} \quad (4.10)$$

which leads to the following first order correction of the constraints

$$\begin{aligned}
T_1^{(1)} &= -2\phi^A\phi^A\eta^1 + 4\phi^A\pi^A\eta^4, \\
T_2^{(1)} &= \eta^3 + 2\pi^A\pi^A\eta^4, \\
T_3^{(1)} &= -\phi^A\phi^A\eta^4, \\
T_4^{(1)} &= -2\lambda\phi^A\phi^A\eta^1 - 2\phi^A\phi^{A''}\eta^1 - 2\phi^A\phi^{A'}\eta^{1'} \\
&\quad - \phi^A\phi^A\eta^{1''} + \eta^2 - 4\phi^{A'}\pi^A\eta^{4'}
\end{aligned} \tag{4.11}$$

It is important to emphasize that the coefficients  $X_{21}$  and  $X_{44}$  could have taken any value. For the former, we choose the simplest possibility and make it zero. For the last one, this would not work. If we had also taken it as zero we would run into difficulties in the second step of the method. The choice we have made was to get a term like  $8\phi^A\phi^{A'}\eta^{4'}\delta$  into eq.(4.3) for  $a = 1$  and  $b = 4$ . This and the term  $8\phi^A\phi^{A'}\eta^4\delta'$ , that appears in the same equation and that could not be matched without spoiling what was already fixed, can be put together as  $8\phi^A\phi^A(\eta^4\delta)'$ . This result makes possible to fix  $T_1^{(2)}$  and  $T_4^{(2)}$ . For details of a similar calculation, see the example discussed in the Appendix.

The combination of (3.10), (4.3), and (4.11) gives

$$\begin{aligned}
&\{-2\phi^A\phi^A\eta^1 + 4\phi^A\pi^A\eta^4, T_2^{(2)}\}_{(\eta)} + \{T_1^{(2)}, \eta^3 + 2\pi^A\pi^A\eta^4\}_{(\eta)} \\
&= 4\phi^A\phi^A\eta^1\delta - 8\phi^A\pi^A\eta^4\delta,
\end{aligned} \tag{4.12}$$

$$\{-2\phi^A\phi^A\eta^1 + 4\phi^A\pi^A\eta^4, T_3^{(2)}\}_{(\eta)} + \{T_1^{(2)}, -\phi^A\phi^A\eta^4\}_{(\eta)} = 0, \tag{4.13}$$

$$\begin{aligned}
&\{-2\phi^A\phi^A\eta^1 + 4\phi^A\pi^A\eta^4, T_4^{(2)}\}_{(\eta)} + \{T_1^{(2)}, -2\lambda\phi^A\phi^A\eta^1 - 2\phi^A\phi^{A''}\eta^1 \\
&\quad - 2\phi^A\phi^{A'}\eta^{1'} - \phi^A\phi^A\eta^{1''} + \eta^2 - 4\phi^{A'}\pi^A\eta^{4'}\}_{(\eta)} \\
&= 8\phi^A\pi^A\eta^1\delta - 8\pi^A\pi^A\eta^4\delta + 8\lambda\phi^A\phi^A\eta^4\delta + 8\phi^A\phi^{A''}\eta^4\delta \\
&\quad + 8\phi^A\phi^{A'}\eta^{4'}\delta + 8\phi^A\phi^{A'}\eta^4\delta' + 4\phi^A\phi^A\eta^4\delta'',
\end{aligned} \tag{4.14}$$

$$\{\eta^3 + 2\pi^A\pi^A\eta^4, T_3^{(2)}\}_{(\eta)} + \{T_2^{(2)}, -\phi^A\phi^A\eta^4\}_{(\eta)} = -2\phi^A\phi^A\eta^4\delta, \tag{4.15}$$

$$\begin{aligned}
&\{\eta^3 + 2\pi^A\pi^A\eta^4, T_4^{(2)}\}_{(\eta)} + \{T_2^{(2)}, -2\lambda\phi^A\phi^A\eta^1 - 2\phi^A\phi^{A''}\eta^1 \\
&\quad - 2\phi^A\phi^{A'}\eta^{1'} - \phi^A\phi^A\eta^{1''} + \eta^2 - 4\phi^{A'}\pi^A\eta^{4'}\}_{(\eta)} \\
&= -4\lambda\phi^A\phi^A\eta^1\delta - 4\phi^A\phi^{A''}\eta^1\delta - 4\phi^A\phi^{A'}\eta^{1'}\delta \\
&\quad - 2\phi^A\phi^A\eta^{1''}\delta + 8\lambda\phi^A\pi^A\eta^4\delta + 8\phi^{A''}\pi^A\eta^4\delta \\
&\quad - 4\phi^{A'}\pi^A\eta^{4'}\delta - 4\phi^A\pi^{A'}\eta^{4'}\delta - 4\phi^A\pi^A\eta^{4''}\delta \\
&\quad - 4\phi^A\phi^{A'}\eta^1\delta' - 2\phi^A\phi^A\eta^{1'}\delta' + 8\phi^{A'}\pi^A\eta^4\delta' \\
&\quad - 4\phi^A\pi^A\eta^{4'}\delta' - 2\phi^A\phi^A\eta^1\delta'' + 4\phi^A\pi^A\eta^4\delta'',
\end{aligned} \tag{4.16}$$

$$\begin{aligned}
&\{-\phi^A\phi^A\eta^4, T_4^{(2)}\}_{(\eta)} + \{T_3^{(2)}, -2\lambda\phi^A\phi^A\eta^1 - 2\phi^A\phi^{A''}\eta^1 \\
&\quad - 2\phi^A\phi^{A'}\eta^{1'} - \phi^A\phi^A\eta^{1''} + \eta^2 - 4\phi^{A'}\pi^A\eta^{4'}\}_{(\eta)} \\
&= -2\phi^A\phi^A\eta^1\delta + 4\phi^A\pi^A\eta^4\delta
\end{aligned} \tag{4.17}$$

After a hard and careful algebraic work, we obtain the following solution for this set of equations

$$\begin{aligned}
T_1^{(2)} &= -8\phi^A\pi^A\eta^1\eta^4 + 4(\pi^A\pi^A - \phi^A\phi^{A''} - \lambda\phi^A\phi^A)\eta^4\eta^4 \\
&\quad - 8\phi^A\phi^{A'}\eta^4\eta^{4'} - 4\phi^A\phi^A\eta^4\eta^{4''} \\
T_2^{(2)} &= -2\eta^1\eta^3 + 4(\phi^A\phi^{A''} + \phi^{A'}\phi^{A'})\eta^1\eta^4 + 4\phi^A\phi^{A'}\eta^{1'}\eta^4 \\
&\quad + 2\phi^A\phi^A\eta^{1''}\eta^4 + 8\phi^A\phi^{A'}\eta^1\eta^{4'} + 2\phi^A\phi^A\eta^{1'}\eta^{4'} \\
&\quad - 4(\lambda\phi^A\pi^A + \phi^{A''}\pi^A)\eta^4\eta^4 + 2\phi^A\pi^A\eta^{4'}\eta^{4'} \\
&\quad - 4\phi^A\pi^A\eta^4\eta^{4''} - 8\phi^{A'}\pi^A\eta^4\eta^{4'} \\
T_3^{(2)} &= 2\phi^A\phi^A\eta^1\eta^4 - 2\phi^A\pi^A\eta^4\eta^4 \\
T_4^{(2)} &= 4(\phi^A\pi^{A'} + 3\phi^{A'}\pi^A)\eta^1\eta^{4'} + 8\phi^A\pi^A\eta^1\eta^{4''}
\end{aligned} \tag{4.18}$$

Contrary to the case without Lagrange multipliers, the quantities which are obtained by summing  $T_a$ ,  $T_a^{(1)}$ , and  $T_a^{(2)}$  are not first-class constraints. It would be necessary to go to higher steps of the method. As one observes this is not an easy task, because there might be other solutions in the second step, besides the one we have found. We do not know which solution would be more appropriate to use in the third step or, more than that, if this solution would actually lead to a solution there.

## 5 Conclusion

We have considered the use of the BFFT formalism for systems with nonlinear constraints. First, we have done a general analysis of the conditions in which the method could be used without going to higher order of the iterative process. In this way, we show why a particular version of the  $O(N)$  nonlinear sigma model can work for this case. We also show that the same occurs for the supersymmetric case, but leading to a nonlocal theory. We have also discussed the general case and show that the full BFFT method is not feasible to be applied. In part because the first iterative step does not give a unique solution and we do not know a priori what is the most convenient one to carry out to the second step. More than that, we have also show that nor all solutions of the first step lead to a solution in the second one.

A way we envisage to circumvent the problems we have mentioned is to consider the method in a less restrictive way, that is to say, by considering that first-class constraints do not have to form just an Abelian algebra, but to be open to the fact that a non-Abelian algebra could be also possible [7].

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## Appendix

One of the unexpected aspect of the BFFT method is that nor all solution obtained in the first step of the method can lead to a solution in the second step. In order to see this point with details, let us consider another solution of eq. (4.9), for example

$$\begin{aligned}
X_{11} &= 0, & X_{12} &= 0, & X_{13} &= 2\delta, & X_{14} &= \phi^A \pi^A \delta, \\
X_{21} &= \phi^A \phi^A \delta, & X_{22} &= 0, & X_{23} &= 0, \\
X_{24} &= \frac{1}{2} \pi^A \pi^A \delta - \frac{1}{2} \phi^A \phi^{A''} \delta - \frac{1}{2} \phi^A \phi^{A'} \delta', \\
X_{31} &= 0, & X_{32} &= 0, & X_{33} &= 0, & X_{34} &= -\frac{1}{4} \phi^A \phi^A \delta, \\
X_{41} &= 0 & X_{42} &= 4\delta, & X_{43} &= \delta'' + 2\lambda\delta, & X_{44} &= 0.
\end{aligned} \tag{A.1}$$

We mention that  $X_{23}$  and  $X_{44}$  could have taken any value. The choice above leads to the following equations in the second step of the method

$$\begin{aligned}
\{2\eta^3 + \phi^A \pi^A \eta^4, T_2^{(2)}\}_{(\eta)} + \{T_1^{(2)}, \phi^A \phi^A \eta^1 - \frac{1}{2} \phi^A \phi^{A''} \eta^4 \\
- \frac{1}{2} \phi^A \phi^{A'} \eta^{4'} + \frac{1}{2} \pi^A \pi^A \eta^4\}_{(\eta)} = -2\phi^A \pi^A \eta^4 \delta,
\end{aligned} \tag{A.2}$$

$$\{2\eta^3 + \phi^A \pi^A \eta^4, T_3^{(2)}\}_{(\eta)} - \frac{1}{4} \phi^A \phi^A \{T_1^{(2)}, \eta^4\}_{(\eta)} = 0, \tag{A.3}$$

$$\begin{aligned}
\{2\eta^3 + \phi^A \pi^A \eta^4, T_4^{(2)}\}_{(\eta)} + \{T_1^{(2)}, 4\eta^2 + \eta^{3''} + 2\lambda\eta^3\}_{(\eta)} \\
= -2\pi^A \pi^A \eta^4 \delta + 2\lambda \phi^A \phi^A \eta^4 \delta + 2\phi^A \phi^{A''} \eta^4 \delta \\
+ 2\phi^A \phi^{A'} \eta^4 \delta' + \phi^A \phi^A \eta^4 \delta'',
\end{aligned} \tag{A.4}$$

$$\begin{aligned}
\{\phi^A \phi^A \eta^1 - \frac{1}{2} \phi^A \phi^{A''} \eta^4 - \frac{1}{2} \phi^A \phi^{A'} \eta^{4'} + \frac{1}{2} \pi^A \pi^A \eta^4, T_3^{(2)}\}_{(\eta)} \\
- \frac{1}{4} \phi^A \phi^A \{T_2^{(2)}, \eta^4\}_{(\eta)} = -\frac{1}{2} \phi^A \phi^A \eta^4 \delta,
\end{aligned} \tag{A.5}$$

$$\begin{aligned}
\{\phi^A \phi^A \eta^1 - \frac{1}{2} \phi^A \phi^{A''} \eta^4 - \frac{1}{2} \phi^A \phi^{A'} \eta^{4'} + \frac{1}{2} \pi^A \pi^A \eta^4, T_4^{(2)}\}_{(\eta)} \\
+ \{T_2^{(2)}, 4\eta^2 + \eta^{3''} + 2\lambda\eta^3\}_{(\eta)} = -4\phi^A \pi^A \eta^1 \delta \\
+ 3\phi^{A''} \pi^A \eta^4 \delta + 2\lambda \phi^A \pi^A \eta^4 \delta + \phi^{A'} \pi^A \eta^{4'} \delta \\
+ \phi^A \pi^{A''} \eta^4 \delta + \phi^A \pi^{A'} \eta^{4'} \delta + 2\phi^A \pi^{A'} \eta^4 \delta'
\end{aligned}$$

$$\begin{aligned}
& + \phi^A \pi^A \eta^{4'} \delta' + 2 \phi^{A'} \pi^A \eta^4 \delta' + 2 \phi^A \pi^A \eta^4 \delta'' , \\
& - \frac{1}{4} \phi^A \phi^A \{ \eta^4, T_4^{(2)} \}_{(\eta)} + \{ T_3^{(2)}, 4\eta^2 + \eta^{3''} + 2\lambda\eta^3 \}_{(\eta)} \\
& = 2\eta^3 \delta + \phi^A \pi^A \eta^4 \delta
\end{aligned} \tag{A.6}$$

$$= 2\eta^3 \delta + \phi^A \pi^A \eta^4 \delta \tag{A.7}$$

Of course, the best strategy to solve these equations is to start from the most complicated one because in the simplest cases there are much freedom in fixing the quantities  $T_a^{(2)}$  and we do not know a priori what would be the best choice we have to do. Let us then consider eq.(A.6). We notice that the right side can only be matched by means of  $T_2^{(2)}$ .

In order to fix the term with  $\lambda$  in the right side of (A.6), we have two options. The first one is  $T_2^{(2)}$  having a term like

$$T_2^{(2)} = \phi^A \pi^A \eta^1 \eta^4 + \dots \tag{A.8}$$

where dots are representing other terms that we shall figure out later. With the choice above, we have

$$\begin{aligned}
\{ T_2^{(2)}, 4\eta^2 + \eta^{3''} + 2\lambda\eta^3 \}_{(\eta)} &= \{ \phi^A \pi^A \eta^1 \eta^4 + \dots, 4\eta^2 + \eta^{3''} + 2\lambda\eta^3 \}_{(\eta)} \\
&= 2\lambda \phi^A \pi^A \eta^4 \delta - 4 \phi^A \pi^A \eta^1 \delta \\
&\quad + \phi^A \pi^A \eta^4 \delta'' + \dots
\end{aligned} \tag{A.9}$$

We notice that the term  $2\lambda \phi^A \pi^A \eta^4 \delta$  was actually obtained, but the term  $\phi^A \pi^A \eta^4 \delta''$  does not match the corresponding one in the right side of (A.6). A solution could be the introduction of a new term in the expression of  $T_2^{(2)}$  like

$$T_2^{(2)} = \phi^A \pi^A \eta^1 \eta^4 - \frac{1}{4} \phi^A \pi^A \eta^4 \eta^{4''} + \dots \tag{A.10}$$

In fact, this new term would give us the term that is lacking,  $\phi^A \pi^A \eta^4 \delta''$ . However, it will also lead to the term  $\phi^A \pi^A \eta^{4''} \delta$ , that does not match any other term in (A.6). We do not have any more options to solve the problems we have.

Let us recall what we have done till now. Looking at eq.(A.6), we had decided to fix the term with  $\lambda$  in the right side, but this did not work. Let us then forget the term in  $\lambda$  and try another solution, for example,

$$T_2^{(2)} = 2 \phi^A \pi^A \eta^1 \eta^4 + \dots \tag{A.11}$$

This choice yields

$$\{ T_2^{(2)}, 4\eta + \eta^{3''} + 2\lambda\eta^3 \}_{(\eta)} = -8 \phi^A \pi^A \eta^1 \delta + 2 \phi^A \pi^A \eta^4 \delta'' + 4\lambda \phi^A \pi^A \eta^4 \delta + \dots \tag{A.12}$$



There are two terms that do not match the right side of (A.6). We may then add another term in (A.11), say

$$T_2^{(2)} = 2\phi^A\pi^A\eta^1\eta^4 + \frac{1}{4}\lambda\phi^A\pi^A\eta^4\eta^4 + \dots \quad (\text{A.13})$$

This leads to

$$\{T_2^{(2)}, 4\eta + \eta^{3''} + 2\lambda\eta^3\}_{(\eta)} = -8\phi^A\pi^A\eta^1\delta + 2\phi^A\pi^A\eta^4\delta'' + 2\lambda\phi^A\pi^A\eta^4\delta + \dots \quad (\text{A.14})$$

However, the first term also does not match the corresponding one into the right side of (A.6). There is no other way we could try in order to solve these problems. So, we cannot find a solution for eq.(A.6) <sup>3</sup>.

We could then think to go back to the solution where we have fixed (A.6) and try to make some refinements on it. For example, we had taken  $X_{44} = 0$ , but we have also seen that it could have taken any other value for it. Let us then conveniently make

$$X_{44} = \delta. \quad (\text{A.15})$$

Thus, the term  $T_4^{(1)}$  turns to be

$$T_4^{(1)} = 4\eta^2 + \eta^{3''} + 2\lambda\eta^3 + \eta^4 \quad (\text{A.16})$$

and consequently, instead of (A.6) we have

$$\begin{aligned} & \{\phi^A\phi^A\eta^1 - \frac{1}{2}\phi^A\phi^{A''}\eta^4 - \frac{1}{2}\phi^A\phi^{A'}\eta^{4'} + \frac{1}{2}\pi^A\pi^A\eta^4, T_4^{(2)}\}_{(\eta)} \\ & + \{T_2^{(2)}, 4\eta^2 + \eta^{3''} + 2\lambda\eta^3 + \eta^4\}_{(\eta)} \\ & = -4\phi^A\pi^A\eta^1\delta + 3\phi^{A''}\pi^A\eta^4\delta + 2\lambda\phi^A\pi^A\eta^4\delta \\ & + \phi^{A'}\pi^A\eta^{4'}\delta + \phi^A\pi^{A''}\eta^4\delta + 2\phi^A\pi^{A'}\eta^4\delta' \\ & + 2\phi^A\pi^A\eta^4\delta'' + \phi^A\pi^{A'}\eta^{4'}\delta + \phi^A\pi^A\eta^{4'}\delta' \\ & + 2\phi^{A'}\pi^A\eta^1\delta' \end{aligned} \quad (\text{A.17})$$

We now consider

$$T_2^{(2)} = \phi^A\pi^A\eta^1\eta^4 + \phi^A\pi^A\eta^{2''}\eta^4 + \dots \quad (\text{A.18})$$

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<sup>3</sup>Even if we were able to obtain the so mentioned terms, there would be more. For example, how could we match the terms  $\phi^{A'}\pi^A\eta^{4'}\delta$  and  $2\phi^{A'}\pi^A\eta^4\delta'$ ? The factor 2 spoils any attempt. The same occurs with  $2\phi^A\pi^{A'}\eta^4\delta'$  and  $\phi^A\pi^{A'}\eta^{4'}\delta$ .

and we then have

$$\begin{aligned} \{T_2^{(2)}, 4\eta^2 + \eta^{3''} + 2\lambda\eta^3 + \eta^4\}_{(\eta)} &= -4\phi^A\pi^A\eta^1\delta \\ &+ 2\phi^A\pi^A\eta^4\delta'' + 2\lambda\phi^A\pi^A\eta^4\delta - 4\phi^A\pi^A\eta^{2''}\delta + \dots \end{aligned} \quad (\text{A.19})$$

We notice that the term  $2\phi^A\pi^A\eta^4\delta''$  was obtained, but the price paid was also the obtainment of the term  $-4\phi^A\pi^A\eta^{2''}\delta$  that does not match any term in the right side of (A.6). There is no other choice for  $X_{44}$  that could be more convenient. Definitely, eq. (A.6) does not have solution.

It is opportune to mention that there are many other attempts that do not work also. Let us write down some of these examples,

$$\begin{aligned} X_{11} &= -2\phi^A\phi^A\delta, & X_{12} &= 0, & X_{13} &= 0, & X_{14} &= 4\phi^A\pi^A\delta, \\ X_{21} &= 0, & X_{22} &= 0, & X_{23} &= \delta, & X_{24} &= 2\pi^A\pi^A\delta, \\ X_{31} &= 0, & X_{32} &= 0, & X_{33} &= 0, & X_{34} &= -\phi^A\phi^A\delta, \\ X_{41} &= -2\lambda\phi^A\phi^A\delta - 2\phi^A\phi^{A''}\delta - 2\phi^A\phi^{A'}\delta' - \phi^A\phi^A\delta'', \\ X_{42} &= \delta, & X_{43} &= 0, & X_{44} &= 0. \end{aligned} \quad (\text{A.20})$$

$$\begin{aligned} X_{11} &= -2\phi^A\phi^A\delta, & X_{12} &= 0, & X_{13} &= 0, & X_{14} &= 4\phi^A\pi^A\delta, \\ X_{21} &= 0, & X_{22} &= 0, & X_{23} &= \delta, \\ X_{24} &= -2\lambda\phi^A\phi^A\delta - 2\phi^A\phi^{A''}\delta - 2\phi^A\phi^{A'}\delta' - \phi^A\phi^A\delta'', \\ X_{31} &= 0, & X_{32} &= 0, & X_{33} &= 0, & X_{34} &= -\phi^A\phi^A\delta, \\ X_{41} &= 2\pi^A\pi^A\delta, & X_{42} &= \delta, & X_{43} &= 0, & X_{44} &= 0. \end{aligned} \quad (\text{A.21})$$

$$\begin{aligned} X_{11} &= 0, & X_{12} &= 0, & X_{13} &= 0, & X_{14} &= \delta, \\ X_{21} &= 0, & X_{22} &= 2\phi^A\phi^A\delta, & X_{24} &= 0, \\ X_{23} &= 2\pi^A\pi^A\delta - 2\lambda\phi^A\phi^A\delta - 2\phi^A\phi^{A''}\delta - 2\phi^A\phi^{A'}\delta' - \phi^A\phi^A\delta'', \\ X_{31} &= 0, & X_{32} &= 0, & X_{33} &= -\phi^A\phi^A\delta, & X_{34} &= 0, \\ X_{41} &= \delta, & X_{42} &= 4\phi^A\pi^A\delta, & X_{43} &= 0, & X_{44} &= 0. \end{aligned} \quad (\text{A.22})$$

$$\begin{aligned} X_{11} &= 0, & X_{12} &= 0, & X_{13} &= 0, & X_{14} &= \delta, \\ X_{21} &= -2\pi^A\pi^A\delta + 2\phi^A\phi^{A''}\delta + 2\phi^A\phi^{A'}\delta', \\ X_{22} &= 2\phi^A\phi^A\delta, & X_{23} &= 0, & X_{24} &= 0, \\ X_{31} &= \phi^A\phi^A\delta, & X_{32} &= 0, & X_{33} &= 0, & X_{34} &= 0, \\ X_{41} &= 0, & X_{42} &= 4\phi^A\pi^A\delta, & X_{43} &= \delta, & X_{44} &= \lambda\delta + \frac{1}{2}\delta''. \end{aligned} \quad (\text{A.23})$$

There are many other solutions, but just corresponding to permutations among the coefficients of the solutions above. It is important to mention that (A.20) corresponds to the case discussed in the text for a different value of the coefficient  $X_{44}$ . The fact that suggest me to concentrate in this solution and not on the others is that it was the first case where we could find a solution for the most complicated of the equations. In all other cases, this did not occur.

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